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Finite-Temperature Entanglement Dynamics in an Anisotropic Two-Qubit Heisenberg Spin Chain

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Abstract This paper investigates the entanglement dynamics of an anisotropic two-qubit Heisenberg spin chain in the presence of decoherence at finite temperature. The time evolution of the concurrence is studied for different initial Werner states. The influences of initial purity, finite temperature, spontaneous decay and Hamiltonian on the entanglement evolution are analyzed in detail. Our calculations show that the finite temperature restricts the evolution of the entanglement all the time when the Hamiltonian improves it and the spontaneous decay to the reservoirs can produce quantum entanglement with the anisotropy of spin-spin interaction. Finally, the steady-state concurrence which may remain non-zero for low temperature is also given.

Keywords Entanglement dynamic · Finite temperature · Werner state · Decoherence

1 Introduction

Quantum entanglement is a property of correlated quantum systems and has played an important role in quantum information theory. In real quantum systems, inevitable interactions with surrounding environment may lead to decoherence resulting in the degradation of entanglement. Yu and Eberly [1, 2] have shown that the entanglement of bipartite qubit system may decrease abruptly to zero in a finite time due to the influence of decoherence, this phenomenon was termed as entanglement sudden death (ESD). Recently, a great deal of the theoretical investigations of ESD have been reported [3–13]. Considering the systems and the reservoirs as a whole, Refs. [14, 15] presented the decay process of the entanglement and pointed out that the lost entanglement was transferred to the freedom degrees of reservoirs via spontaneous decay. Furthermore, A. Al-Qasimi and D F.V. James found that the ESD regarding two-qubit system would always appear in any finite-temperature

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reservoirs if ignored the action of Hamiltonian [16]. However, many investigations showed that the entanglement exists naturally in the two-qubit Heisenberg spin 1/2 system owing to the spin exchange interaction [17, 18]. Other interactions, such as the spin-orbit coupling Dzyaloshinskii-Moriya (DM) interaction [19–21] and the external magnetic field, can improve the entanglement of the quantum system [22–24]. The entanglement sudden birth (ESB), which is the creation or rebirth of entanglement where the initial unentangled qubits can be entangled after a finite evolution time [15, 25, 26], may arise from the interactions. Therefore, it is necessary to include the Hamiltonian in the studies of entanglement dynamics with decoherence at finite temperature.

In this paper, we present an exact calculation of the entanglement dynamics between two qubits in finite-temperature reservoirs. Based on Ref. [16], we consider the Hamiltonian of an anisotropic two-qubit Heisenberg spin chain with DM interaction and assume the initial states to be Werner states. We pay attention to the effects of the system parameters. It is found that the ESD and ESB are sensitive to the mixed purity. The temperature can accelerate the ESD, so the zero temperature brings the best evolution of the entanglement. The DM coupling interaction and anisotropic spin exchange interaction can influence the entanglement dynamics through the Hamiltonian. Especially, the reservoirs can cause the entanglement by combining the anisotropy spin-spin interaction of qubits. The value of steady-state entanglement depends on the finite temperature, the decay rate to the reservoirs and the anisotropy exchange interaction between qubits.

2 Two-Qubit Hamiltonian

The entanglement of the two-qubit system must be represented by the density matrix ρ, and ρ will change in time when the system is exposed to quantum noise. We study a two-qubit system in a heat bath, and the qubits interact with their reservoirs respectively. The description of the time evolution of an open system at finite temperature is provided by the master equation, which can be written most generally in the Lindblad form with the assumption of weak system-reservoir coupling and Born-Markov approximation [27]. The Lindblad equation for the density matrix ρ is given by

$$\frac{d\rho}{dt} = -\mathrm{i}[H,\rho] + \sum_{j=1,2}\left[(n+1)\gamma_j\left(\sigma_{j-}\rho\sigma_{j+} - \frac{1}{2}\{\sigma_{j+}\sigma_{j-},\rho\}\right)\right.$$
$$\left. + n\gamma_j\left(\sigma_{j+}\rho\sigma_{j-} - \frac{1}{2}\{\sigma_{j-}\sigma_{j+},\rho\}\right)\right], \tag{1}$$

where the term γ_1 (γ_2) is the spontaneous decay rate of qubit 1 (2) to the reservoir, and here we assume $\gamma_1 = \gamma_2 = \gamma$; $\{\}$ means anticommutator. $\sigma_\pm = \frac{1}{2}(\sigma_x \pm \mathrm{i}\sigma_y)$ are the spin raising and lowering operators, and σ_i $(i = x, y, z)$ are the Pauli matrices. n is the average thermal excitation of the reservoirs, denotes the temperature. $n = 0$ indicates the zero temperature and finite n corresponds to the finite temperature.

We consider an anisotropic two spin-$\frac{1}{2}$ particles Heisenberg XYZ chain coupled to the reservoirs with only nearest-neighbor interactions and the DM interaction along z-axis. The corresponding Hamiltonian reads

$$H = \frac{1}{2}[J_x\sigma_{1x}\sigma_{2x} + J_y\sigma_{1y}\sigma_{2y} + J_z\sigma_{1z}\sigma_{2z} + \overrightarrow{D}\cdot(\overrightarrow{\sigma_1}\times\overrightarrow{\sigma_2})], \tag{2}$$

where J_i $(i = x, y, z)$ are the spin-spin coupling coefficients, $J_i > 0$ corresponds to the antiferromagnetic case, and $J_i < 0$ corresponds to the ferromagnetic case. $\overrightarrow{D}$ is called DM vector coupling, here we choose $\overrightarrow{D} = D\overrightarrow{z}$. The Hamiltonian can be expressed as

$$H = (J + \mathrm{i}D)\sigma_{1+}\sigma_{2-} + (J - \mathrm{i}D)\sigma_{1-}\sigma_{2+} + \Delta(\sigma_{1+}\sigma_{2+} + \sigma_{1-}\sigma_{2-}) + \frac{J_z}{2}\sigma_{1z}\sigma_{2z}, \tag{3}$$

where $J = \frac{J_x+J_y}{2}$, $\Delta = \frac{J_x-J_y}{2}$, the parameter Δ describes the spatial anisotropy of the spin-spin interaction. Under the most situations, we consider $J = 1$.

We assume that the system initial states are in Werner states [28, 29], and according to the Hamiltonian, the density matrix $\rho(t)$ can be represented as

$$\rho(t) = \begin{pmatrix} \rho_{11}(t) & 0 & 0 & \rho_{14}(t) \\ 0 & \rho_{22}(t) & \rho_{23}(t) & 0 \\ 0 & \rho_{32}(t) & \rho_{33}(t) & 0 \\ \rho_{41}(t) & 0 & 0 & \rho_{44}(t) \end{pmatrix}. \tag{4}$$

All other matrix elements are zero. Combining (1), (3) and (4), we obtain the first-order differential equations of the density matrix elements in the standard basis $\{|11\rangle, |10\rangle, |01\rangle, |00\rangle\}$:

$$\begin{aligned}
\frac{d\rho_{11}(t)}{dt} &= -2(1+n)\gamma\rho_{11}(t) + n\gamma[\rho_{22}(t) + \rho_{33}(t)] - \mathrm{i}\Delta[\rho_{41}(t) - \rho_{14}(t)], \\
\frac{d\rho_{14}(t)}{dt} &= -2(1+2n)\gamma\rho_{14}(t) - \mathrm{i}\Delta[\rho_{44}(t) - \rho_{11}(t)], \\
\frac{d\rho_{22}(t)}{dt} &= (1+n)\gamma[\rho_{11}(t) - \rho_{22}(t)] + n\gamma[\rho_{44}(t) - \rho_{22}(t)] \\
&\quad - \mathrm{i}J[\rho_{32}(t) - \rho_{23}(t)] + D[\rho_{32}(t) + \rho_{23}(t)], \\
\frac{d\rho_{23}(t)}{dt} &= -\mathrm{i}J[\rho_{33}(t) - \rho_{22}(t)] + D[\rho_{33}(t) - \rho_{22}(t)] - (1+2n)\gamma\rho_{23}(t), \\
\frac{d\rho_{33}(t)}{dt} &= (1+n)\gamma[\rho_{11}(t) - \rho_{33}(t)] + n\gamma[\rho_{44}(t) - \rho_{33}(t)] \\
&\quad - \mathrm{i}J[\rho_{23}(t) - \rho_{32}(t)] - D[\rho_{23}(t) + \rho_{32}(t)], \\
\frac{d\rho_{44}(t)}{dt} &= (1+n)\gamma[\rho_{22}(t) + \rho_{33}(t)] + 2n\gamma\rho_{44}(t) - \mathrm{i}\Delta[\rho_{14}(t) - \rho_{41}(t)],
\end{aligned} \tag{5}$$

where $\rho_{41}(t) = \rho_{14}^*(t)$, $\rho_{32}(t) = \rho_{23}^*(t)$. From above, we know that the coupling coefficient J_z is useless to the entanglement evolution. The solutions of (5) depend on the initial states of the qubits, we choose the initial states $\rho_0 = \frac{1-p}{4}I_4 + p|\Phi\rangle\langle\Phi|$ and $\rho_0' = \frac{1-q}{4}I_4 + q|\Psi\rangle\langle\Psi|$, in which $|\Phi\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle$ and $|\Psi\rangle = \cos\phi|01\rangle + \sin\phi|10\rangle$ with $|1\rangle$ denoting spin-up, and $|0\rangle$ symbolling spin-down. The entanglement of the initial states is measured by the purity p or q and the mixing θ or ϕ.

3 Entanglement Dynamics at Finite Temperature

In order to investigate the entanglement dynamics of the two-qubit system, we will employ the concurrence to assess entanglement [30, 31]. For the system described by the density

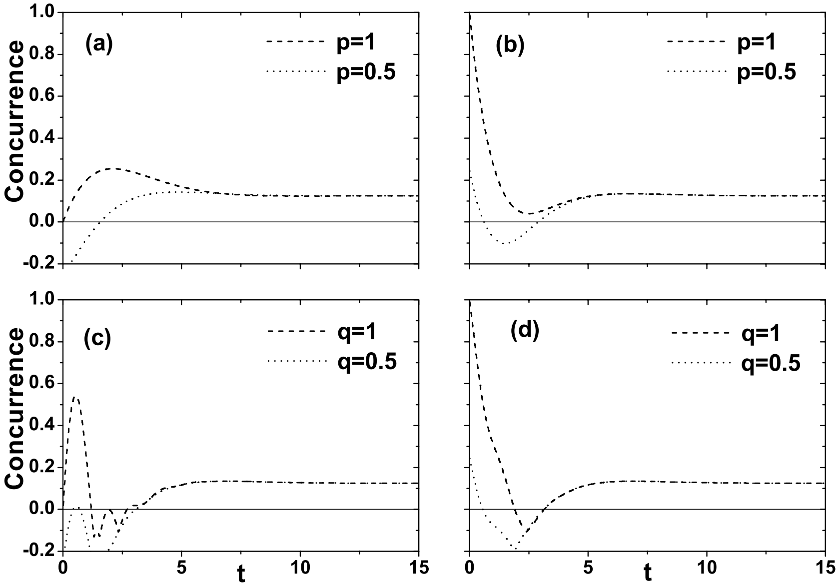

Fig. 1 The concurrence $C(t)$ is plotted as a function of time t and purity p or q with decay rate $\gamma = 0.5$ and anisotropy $\Delta = 0.2$ at zero temperature for different cases: (**a**) $\theta = 0$, (**b**) $\theta = \pi/4$, (**c**) $J = 1$, $D = 0.5$, $\phi = 0$, (**d**) $J = 1$, $D = 0.5$, $\phi = \pi/4$

matrix ρ, the concurrence is defined by

$$C = \max(\sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}, 0), \tag{6}$$

where λ_1, λ_2, λ_3 and λ_4 are the eigenvalues in a decreasing order of the operator R defined by $R = \rho(\sigma_{1y} \otimes \sigma_{2y})\rho^*(\sigma_{1y} \otimes \sigma_{2y})$, $*$ denotes the complex conjugate, σ_y is the usual Pauli matrix. For the density matrices like (4), the concurrence has a simple form

$$C(t) = \max[2(|\rho_{23}(t)| - \sqrt{\rho_{11}(t)\rho_{44}(t)}), 2(|\rho_{14}(t)| - \sqrt{\rho_{22}(t)\rho_{33}(t)}), 0]. \tag{7}$$

The concurrence is available, no matter whether $\rho(t)$ is pure or mixed. It ranges from 0 to 1, $C(t) = 0$ and $C(t) = 1$ indicate the vanishing entanglement and the maximal entanglement respectively.

In the following, we investigate the entanglement dynamics when the initial states are ρ_0 and ρ_0'. We discuss the ESD and ESB under different system parameters, such as purity p or q, finite temperature of the environment n, spontaneous decay rate to the reservoirs γ, anisotropic parameter Δ and DM coupling coefficient D. Especially, it is found that the entanglement dynamics are independent of DM coupling interaction and the exchange parameter J when the initial quantum state is ρ_0. This phenomenon is related to the Hamiltonian of (3).

As shown in Fig. 1, the time evolution of the concurrence is plotted for various initial states with different purity parameters p or q at zero temperature. From Figs. 1(b) and 1(d), one can find that the ESD is sensitive to the initial states. For the mixed initial states $p < 1$ or $q < 1$, ESD happens readily. Figures 1(a) and 1(c) show that the ESB is swift and clear for the pure states $p = 1$ or $q = 1$. It seems that initial pure states are good for the ESB and may delay the ESD. Another character revealed by Fig. 1 is that various initial states

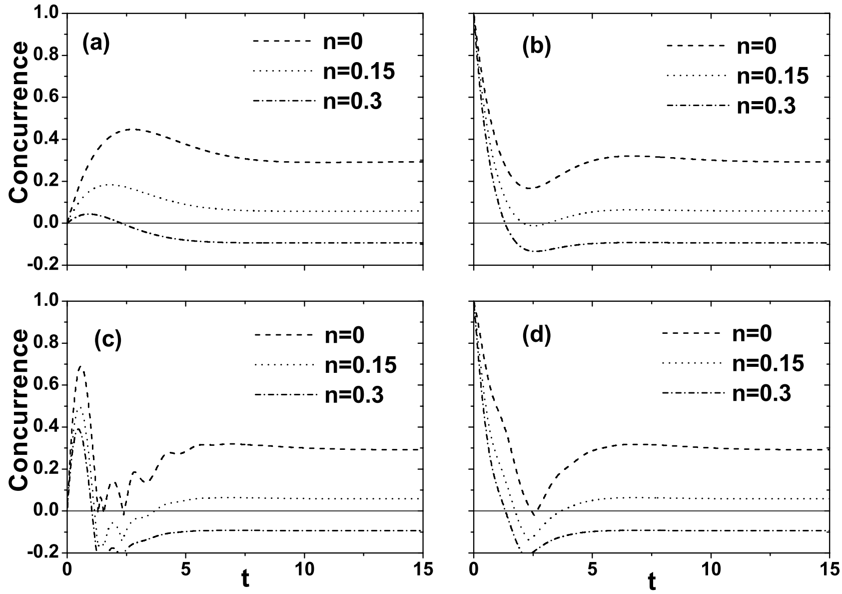

Fig. 2 The concurrence $C(t)$ is plotted as a function of time t and average temperature n with decay rate $\gamma = 0.5$, anisotropy $\Delta = 0.2$ and purity $p = q = 1$ for different cases: (**a**) $\theta = 0$, (**b**) $\theta = \pi/4$, (**c**) $J = 1$, $D = 0.5$, $\phi = 0$, (**d**) $J = 1$, $D = 0.5$, $\phi = \pi/4$

generate the same stable concurrence with the equal interaction parameters. For simplicity in our treatment, we choose $p = 1$ and $q = 1$ in the following.

For the system in thermal equilibrium, the concurrence is largest at zero temperature [17, 18]. The lower temperature is, the stronger entanglement is. Figure 2 presents the influence of finite temperature on the entanglement evolution for several different initial states: unentangled states ($\theta = 0$ or $\phi = 0$), maximal entangled states ($\theta = \pi/4$ or $\phi = \pi/4$). It is shown that the entanglement reduces with increasing temperature. Figures 2(a) and (c) perform that the ESB is more notable at low temperature for initial pure states $|00\rangle$ or $|01\rangle$. The initial states of Figs. 2(b) and (d) are $|00\rangle + |11\rangle$ or $|01\rangle + |10\rangle$. The ESD appears rapidly with high temperature. Moreover, Fig. 2 notes that the higher temperature produces lower steady concurrence. There is no ESB expect ESD at zero temperature without the interactions of Hamiltonian as shown in Fig. 3. Obviously all entanglement vanish in a finite time with decoherence if the Hamiltonian is disregard [16]. Thus it can be seen that the Hamiltonian can improve the entanglement dynamics.

Entanglement, which is the quantum non-local connection, will be unavoidably damaged due to the interaction with the environment. The energy of the qubits is lost via spontaneous decay to the environment, and the reservoirs can lead to excitation of the qubits by combining the interaction of qubits. Figure 4 illustrates the time evolution of the concurrence for different spontaneous decay rates γ. From Figs. 4(a) and (c), one can find that the entanglement evolves periodically without the decoherence ($\gamma = 0$). The ESB is exciting due to the action of the Hamiltonian. The frequencies of the oscillations depend on the anisotropic parameter Δ or DM coupling coefficient D. In the presence of the decoherence, the ESB occurs more rapidly with lower decay rate γ. From the Figs. 4(b) and (d), without the decoherence, the concurrence is unchanged with the initial state $|00\rangle + |11\rangle$ when the entanglement evolves periodically because of the DM interaction D with the initial state $|01\rangle + |10\rangle$.

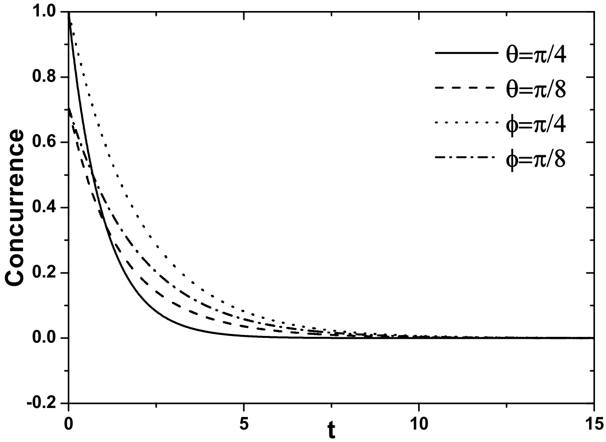

Fig. 3 The concurrence $C(t)$ is plotted as a function of time t for different θ and ϕ with average temperature $n=0$, decay rate $\gamma=0.5$, anisotropy $\Delta=0$, coupling parameter $J=0$, DM interaction $D=0$ and purity $p=q=1$

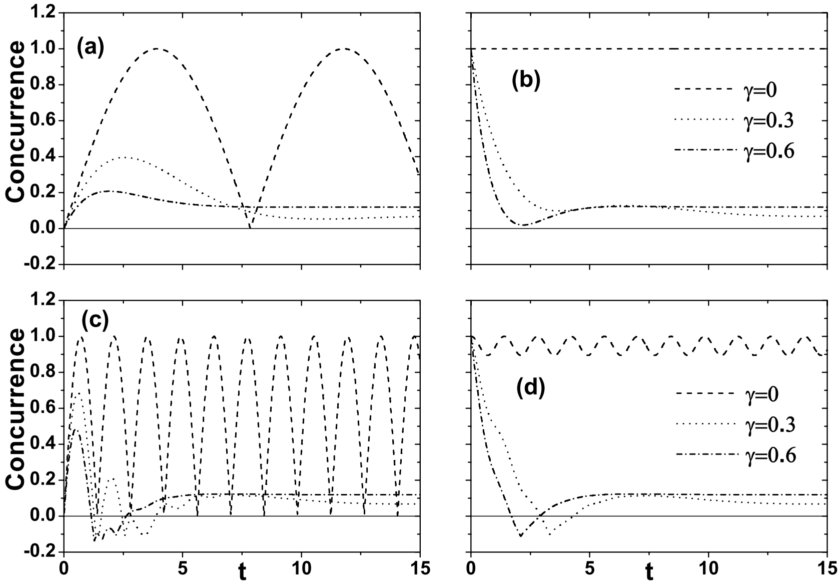

Fig. 4 The concurrence $C(t)$ is plotted as a function of time t and decay rate γ with anisotropy $\Delta=0.2$, average temperature $n=0.1$ and purity $p=q=1$ for different cases: (**a**) $\theta=0$, (**b**) $\theta=\pi/4$, (**c**) $J=1$, $D=0.5$, $\phi=0$, (**d**) $J=1$, $D=0.5$, $\phi=\pi/4$

Taking the decoherence into account, the ESD prefers a higher decay rate γ. From Fig. 4, different decay rates γ result in different steady entanglement. When the decoherence is inevitable, the appropriate value of γ is decided by the Hamiltonian.

As mentioned above, obtained from (3), the DM interaction D and the exchange parameter J cannot act on the entanglement dynamics if the initial state is $\rho_0=\frac{1-p}{4}I_4+p\left|\Phi\right\rangle\left\langle\Phi\right|$. However, the anisotropic parameter Δ can effect the entanglement evolution even if the initial state is $\rho_0'=\frac{1-q}{4}I_4+q\left|\Psi\right\rangle\left\langle\Psi\right|$. Figure 5 describes the influence of anisotropic parameter Δ on the time evolution of the concurrence, different Δ results in various steady entanglement. Figure 5(a) shows that the strength of Δ can improve the ESB with the initial

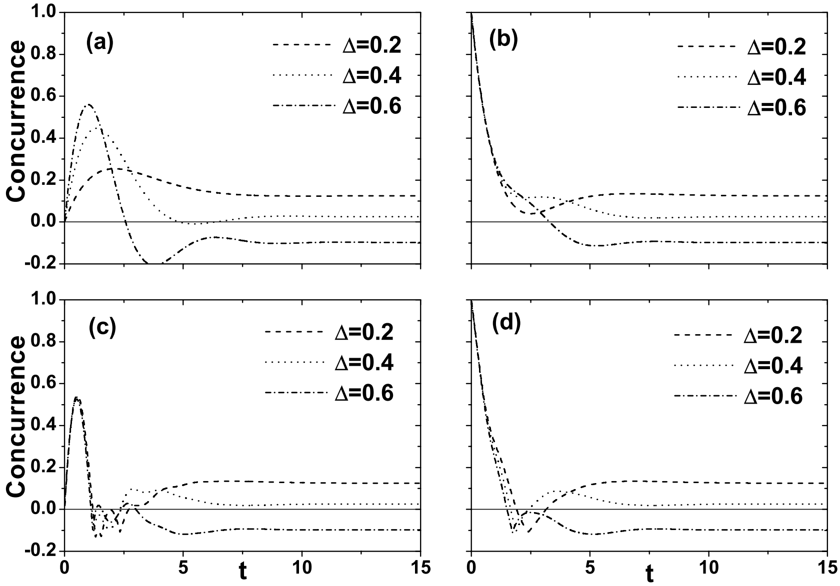

Fig. 5 The concurrence $C(t)$ is plotted as a function of time t and anisotropy Δ with decay rate $\gamma = 0.5$, average temperature $n = 0.1$ and purity $p = q = 1$ for different cases: (**a**) $\theta = 0$, (**b**) $\theta = \pi/4$, (**c**) $J = 1$, $D = 0.5$, $\phi = 0$, (**d**) $J = 1$, $D = 0.5$, $\phi = \pi/4$

state $|00\rangle$. Figure 5(c) reveals the active influence of Δ on the ESB when the initial state is $|01\rangle$. The first ESB is unconcerned with Δ, but the second ESB is in connection with it. In fact, this property generates from the combination of anisotropic parameter Δ and decay rate γ. Figure 6 analyzes their combined action on ESB with initial state $|01\rangle$. The Δ has an active interaction on the concurrence and could bring the ESB only if $\gamma \neq 0$. If the decay rate $\gamma = 0$, the Δ cannot influence the dynamics of entanglement when the initial state is ρ_0'. The DM interaction just causes an oscillation around the evolution of the concurrence. The frequencies of the oscillations are determined by the size of D. In Fig. 7, we plot the time evolution of the concurrence for various values of the DM interaction D. The frequencies of the oscillations increase with the increasing D. On the whole, the decay rate and the anisotropic parameter play an important role in entanglement evolution regardless of the initial states of the system.

From the above discussion, we see that even in the presence of decoherence and finite temperature, the concurrence can still reach a steady value after some oscillatory behavior for a given set of system parameters [32]. According to (5), the corresponding steady concurrence is as followed:

$$C_{steady} = \frac{2\Delta\gamma - 2(n^2+n)\gamma^2 - 2\Delta^2}{4\Delta^2 + (1+2n^2)\gamma^2}. \tag{8}$$

The stable concurrence reduces simply with the increasing finite temperature n, and depends on the combination of anisotropic parameter Δ and decay rate γ.

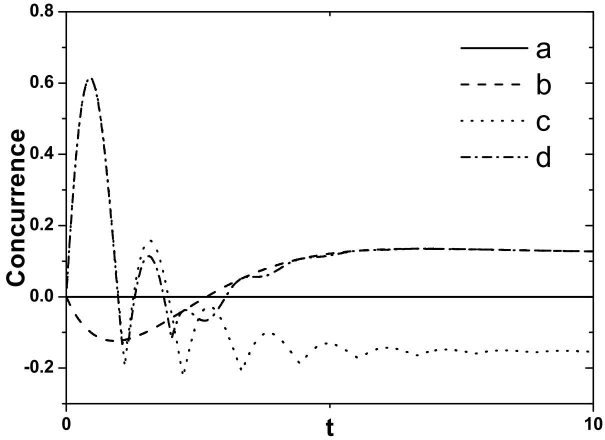

Fig. 6 The concurrence $C(t)$ is plotted as a function of time t with average temperature $n = 0.1$, purity $q = 1$ and mixing $\phi = 0$. The straight, dashed, dotted and dash-dotted lines are a: $J = 0$, $D = 0$, $\Delta = 0.2$, $\gamma = 0$, b: $J = 0$, $D = 0$, $\Delta = 0.2$, $\gamma = 0.5$, c: $J = 1$, $D = 1$, $\Delta = 0$, $\gamma = 0.5$, and d: $J = 1$, $D = 1$, $\Delta = 0.2$, $\gamma = 0.5$

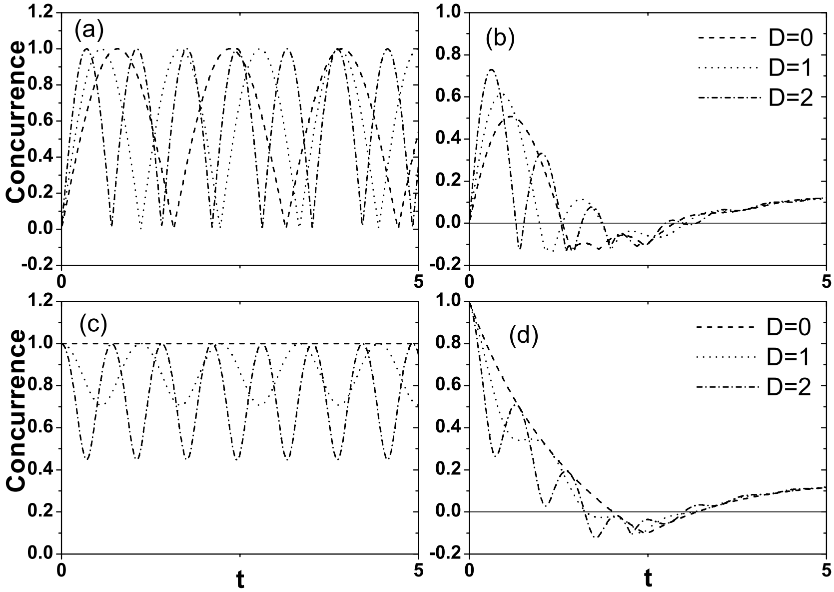

Fig. 7 The concurrence $C(t)$ is plotted as a function of time t and DM interaction D with anisotropy $\Delta = 0.2$, coupling parameter $J = 1$, average temperature $n = 0.1$ and purity $p = q = 1$ for different cases: (**a**) $\gamma = 0$, $\phi = 0$, (**b**) $\gamma = 0.5$, $\phi = 0$, (**c**) $\gamma = 0$, $\phi = \pi/4$, (**d**) $\gamma = 0.5$, $\phi = \pi/4$

4 Conclusions

In conclusion, we present a solution for the entanglement dynamics of a two-qubit anisotropic Heisenberg XYZ chain with different initial system states at finite temperature and investigate the influences of the system parameters. It is found that when initial states are pure, the ESB is forceful and the ESD can be delayed. The increasing finite temperature just bring down the evolution of the entanglement at all times. Taking the interactions of Hamiltonian into account, we find that the evolution of the concurrence is irrelevant to DM coupling interaction with the initial quantum state $\rho_0 = \frac{1-p}{4} I_4 + p\,|\Phi\rangle\,\langle\Phi|$. When the quantum state are initially in $\rho_0' = \frac{1-q}{4} I_4 + q \left|\Psi\right\rangle \left\langle\Psi\right|$, anisotropic parameter can influence the entanglement dynamics only if there is the spontaneous decay to the reservoirs. It is shown that the interactions of the qubits can effect the entanglement dynamics positively. Finally, we take the infinite time limit and obtain a stable entanglement which is controlled by the finite temperature, anisotropic parameter and decay rate. It seems that the stable entanglement is the balance between the anisotropic parameter and the decay rate at finite temperature. Our results will shed light on the understanding of entanglement dynamics of quantum system with environmental effect as well as the ESD and ESB.

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